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A Thick Plate Problem in the Theory of Generalized Thermoelastic Diffusion

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Abstract In this work, the problem of a thermoelastic thick plate with a permeating substance in contact with one of the bounding planes is considered in the context of the theory of generalized thermoelastic diffusion with one relaxation time. The bounding surface of the half-space is taken to be traction free and is subjected to a time-dependent thermal shock. The chemical potential is also assumed to be a known function of time on the bounding plane. Laplace transform techniques are used. The solution is obtained in the Laplace transform domain by using a direct approach. The solution of the problem in the physical domain is obtained numerically using a numerical method for the inversion of the Laplace transform based on Fourier expansion techniques. The temperature, displacement, stress, and concentration as well as the chemical potential are obtained. Numerical computations are carried out and represented graphically.

Keywords Generalized thermoelasticity · Thermal shock · Thermoelastic diffusion

1 Introduction

Biot [1] developed the coupled theory of thermoelasticity to deal with a defect of the uncoupled theory that mechanical causes have no effect on the temperature. However, this theory shares a defect of the uncoupled theory in that it predicts infinite speeds of propagation for heat waves.

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Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended by Sherief [3] and by Dhaliwal and Sherief [4] to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and coupled theories of thermoelasticity. For this theory, Ignaczak [5] studied uniqueness of solution; Sherief [6] proved uniqueness and stability. Anwar and Sherief [7] and Sherief [8] developed the state-space approach to this theory. Anwar and Sherief [9] completed the integral equation formulation. Sherief and Hamza [10,11] solved some two-dimensional problems and studied wave propagation. Sherief and El-Maghraby [12,13] solved two crack problems. El-Maghraby [14–16] solved some two-dimensional problems for media affected by heat sources and body forces.

Diffusion can be defined as the random walk, of an ensemble of particles, from regions of high concentration to regions of lower concentration. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, and form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction.

Nowacki [17–20] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Recently, Sherief et al. [21] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves.

2 Formulation of the Problem

We consider the problem of an isotropic thermoelastic half-space ($x \geq 0$) with a permeating substance (such as a gas) in contact with the upper plane of the half-space ($x = 0$). The x-axis is taken perpendicular to the upper plane pointing inwards. This upper plane of the half-space is taken to be traction free and is subjected to a time-dependent thermal shock. The chemical potential is also assumed to be a known function of time on the upper plane. All considered functions are assumed to be bounded and vanish as $x \to \infty$.

The equation of motion in the absence of body forces is given by [21]

$$\rho\, \ddot{u}_i = \mu\, u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \beta_1\, T_{,i} - \beta_2\, C_{,i}, \tag{1}$$

where u_i are the components of the displacement vector, T is the absolute temperature, C is the concentration of the diffusive material in the elastic body, λ, μ are Lamé's constants, ρ is the density, and β_1 and β_2 are the material constants given by

$$\beta_1 = (3\lambda + 2\mu)\alpha_\mathrm{t} \quad \text{and} \quad \beta_2 = (3\lambda + 2\mu)\alpha_\mathrm{c},$$

α_t is the coefficient of linear thermal expansion, and α_c is the coefficient of linear diffusion expansion.

The energy equation has the form [21],

$$kT_{,ii} = \rho c_E(\dot{T} + \tau_0 \ddot{T}) + \beta_1 T_0(\dot{e}_{kk} + \tau_0 \ddot{e}_{kk}) + aT_0(\dot{C} + \tau_0 \ddot{C}), \tag{2}$$

where k is the thermal conductivity, c_E is the specific heat at constant strain, τ_0 is the thermal relaxation time, 'a' is a measure of the thermodiffusion effect, T_0 is a reference temperature assumed to obey the inequality $|(T - T_0)/T_0| << 1$, and e_{ij} represents the components of the strain tensor given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{3}$$

The diffusion equation has the form [21],

$$d\beta_2\, e_{kk,ii} + daT_{,ii} + \dot{C} + \tau\ddot{C} - db\, C_{,ii} = 0, \tag{4}$$

where d is the diffusion coefficient, b is a measure of the diffusive effect, and τ is the diffusion relaxation time.

The constitutive equations have the form [21],

$$\sigma_{ij} = 2\mu\, e_{ij} + \delta_{ij}\left[\lambda e_{kk} - \beta_1(T - T_0) - \beta_2\, C\right], \tag{5a}$$
$$P = -\beta_2\, e_{kk} + bC - a(T - T_0), \tag{5b}$$

where σ_{ij} are the components of the stress tensor and P is the chemical potential.

It follows from the description of the problem that all considered functions will depend on x and t only. We, thus, obtain the displacement components of the form,

$$u_x = u(x,t), u_y = u_z = 0. \tag{6}$$

The strain components are given by

$$e_{xx} = \mathrm{D}u, e_{yy} = e_{zz} = e_{xy} = e_{yz} = e_{zx} = 0,$$

where $\mathrm{D} = \frac{\partial}{\partial x}$.

The cubical dilatation $e = e_{kk}$ is equal to

$$e = \mathrm{D}u. \tag{7}$$

From Eq. 5a, it follows that the stress tensor components have the form,

$$\sigma = \sigma_{xx} = (\lambda + 2\mu)\mathrm{D}u - \beta_1(T - T_0) - \beta_2 C, \tag{8a}$$

$$\sigma_{yy} = \sigma_{zz} = \lambda \mathrm{D}u - \beta_1(T - T_0) - \beta_2 C, \tag{8b}$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0. \tag{9}$$

Equations 1, 2, and 4, thus, reduce to

$$\rho\ddot{u} = \mu \mathrm{D}^2 u + (\lambda + \mu)\,\mathrm{D}e - \beta_1\,\mathrm{D}T - \beta_2\,\mathrm{D}C, \tag{10}$$

$$k\mathrm{D}^2 T = \rho c_E \left(\dot{T} + \tau_0 \ddot{T}\right) + \beta_1 T_0 \left(\mathrm{D}\dot{u} + \tau_0 \mathrm{D}\ddot{u}\right) + aT_0 \left(\dot{C} + \tau_0 \ddot{C}\right) \tag{11}$$

$$d\,\beta_2\,\mathrm{D}^2 e + da\mathrm{D}^2 T + \dot{C} + \tau\ddot{C} - db\mathrm{D}^2 C = 0. \tag{12}$$

By using Eq. 7, Eqs. 10–12 can be written as

$$\rho\ddot{u} = (\lambda + 2\mu)\mathrm{D}e - \beta_1\,\mathrm{D}T - \beta_2\,\mathrm{D}C, \tag{13}$$

$$k\mathrm{D}^2 T = \rho\, c_E \left(\dot{T} + \tau_0 \ddot{T}\right) + \beta_1 T_0 \left(\dot{e} + \tau_0 \ddot{e}\right) + aT_0 \left(\dot{C} + \tau_0 \ddot{C}\right), \tag{14}$$

$$d\,\beta_2\,\mathrm{D}^2 e + da\mathrm{D}^2 T + \dot{C} + \tau\ddot{C} - db\mathrm{D}^2 C = 0. \tag{15}$$

The governing equations can be put in a more convenient form by using the following dimensionless variables:

$$x^* = c_1 \eta\, x,\, u^* = c_1 \eta\, u,\, t^* = c_1^2 \eta\, t,\, \tau_0^* = c_1^2 \eta\, \tau_0,\, \tau^* = c_1^2 \eta\, \tau,$$
$$\theta^* = \frac{\beta_1(T - T_0)}{\lambda + 2\mu},\, C^* = \frac{\beta_2 C}{\lambda + 2\mu},\, P^* = \frac{P}{\beta_2},\, \sigma_{ij}^* = \frac{\sigma_{ij}}{\lambda + 2\mu},$$

where $c_1^2 = (\lambda + 2\mu)/\rho$, $\eta = \rho c_E/k$.

Using the above dimensionless variables, Eqs. 13–15 take the following form where we have dropped the asterisks for convenience:

$$\ddot{u} = \mathrm{D}^2 u - \mathrm{D}\,\theta - \mathrm{D}\,C, \tag{16}$$

$$\mathrm{D}^2\theta = \dot{\theta} + \tau_0\ddot{\theta} + \varepsilon\,\dot{e} + \tau_0\ddot{e} + \varepsilon\,\alpha_1 \left(\dot{C} + \tau_0 \ddot{C}\right), \tag{17}$$

$$\mathrm{D}^2 e + \alpha_1 \mathrm{D}^2\theta + \alpha_2 \left(\dot{C} + \tau\,\ddot{C}\right) - \alpha_3 \mathrm{D}^2\, C = 0, \tag{18}$$

where $\varepsilon = \frac{\beta_1^2 T_0}{\rho c_E(\lambda+2\mu)}$, $\alpha_1 = \frac{a(\lambda+2\mu)}{\beta_1\beta_2}$, $\alpha_2 = \frac{\lambda+2\,\mu}{\beta_2^2 d\,\eta}$, $\alpha_3 = \frac{b(\lambda+2\mu)}{\beta_2^2}$.

Also, Eqs. 5b and 8a,b take the form,

$$\sigma_{xx} = e - \theta - C, \tag{19a}$$

$$\sigma_{yy} = \sigma_{zz} = \left(1 - 2/\beta^2\right) e - \theta - C, \tag{19b}$$

$$P = \alpha_3 C - e - \alpha_1\theta, \tag{20}$$

where $\beta^2 = (\lambda + 2\mu)/\mu$.

The initial conditions of the problem are taken to be homogeneous while the boundary conditions are assumed to be

$$\sigma(x,t)\,|_{x=h} = 0,\, u(x,t)\,|_{x=-h} = 0, \tag{21}$$

$$\theta(x,t)\,|_{x=h} = f_1(t),\, \frac{\partial\,\theta(x,t)}{\partial\,x}\,|_{x=-h} = 0, \tag{22}$$

$$P(x,t)\,|_{x=h} = f_2(t),\, \frac{\partial\,C(x,t)}{\partial\,x}\,|_{x=-h} = 0, \tag{23}$$

where $f_1(t)$ and $f_2(t)$ are known functions of t. This means that the lower surface is laid on a rigid foundation that is thermally insulated and impermeable while the upper surface is traction free and acted upon by two shocks.

3 Solution in the Laplace Transform Domain

Introducing the Laplace transform defined by the formula,

$$\bar{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t,$$

into Eqs. 16–20 and using the homogeneous initial conditions, we obtain

$$s^2\,\bar{u} = \mathrm{D}^2\,\bar{u} - \mathrm{D}\,\bar{\theta} - \mathrm{D}\,\bar{C}, \tag{24}$$

$$\mathrm{D}^2\,\bar{\theta} = \left(s + \tau_0 s^2\right)\left[\bar{\theta} + \varepsilon\,\bar{e} + \varepsilon\,\alpha_1\bar{C}\right], \tag{25}$$

$$\mathrm{D}^2\,\bar{e} + \alpha_1 \mathrm{D}^2\,\bar{\theta} + \left[\alpha_2\left(s + \tau\,s^2\right) - \alpha_3 \mathrm{D}^2\right]\bar{C} = 0, \tag{26}$$

$$\bar{\sigma}_{xx} = \bar{e} - \bar{\theta} - \bar{C}, \tag{27a}$$

$$\bar{\sigma}_{yy} = \bar{\sigma}_{zz} = \left(1 - 2/\beta^2\right)\bar{e} - \bar{\theta} - \bar{C}, \tag{27b}$$

$$\bar{P} = \alpha_3\bar{C} - \bar{e} - \alpha_1\bar{\theta}. \tag{28}$$

Taking the divergence of Eq. 24, we obtain

$$(\mathrm{D}^2 - s^2)\bar{e} - \mathrm{D}^2\bar{\theta} - \mathrm{D}^2\bar{C} = 0. \tag{29}$$

Eliminating $\bar{e}$ and $\bar{C}$ among Eqs. 25, 26, and 29, we obtain

$$(\mathrm{D}^6 - a_1\,\mathrm{D}^4 + a_2\,\mathrm{D}^2 - a_3)\bar{\theta} = 0, \tag{30}$$

where

$$a_1 = \frac{s}{\alpha_3 - 1}\,[(1 + \tau_0 s)\,(\alpha_1\,\varepsilon\,(\alpha_1 + 2)\, + \alpha_3(\varepsilon + 1) - 1) + \alpha_2(1 + \tau s) + \alpha_3 s]\,,$$
$$a_2 = \frac{s^2}{\alpha_3 - 1}\,\left[(1 + \tau_0 s)\left(\varepsilon\,s\,\alpha_1^2 + \alpha_3 s + \alpha_2(\varepsilon + 1)(1 + \tau s)\right) + \alpha_2 s(1 + \tau s)\right],$$
$$a_3 = \frac{s^4\,\alpha_2}{\alpha_3 - 1}(1 + \tau\,s)(1 + \tau_0 s).$$

In a similar manner, we can show that $\bar{e}$ and $\bar{C}$ satisfy the equations,

$$(\mathrm{D}^6 - a_1\,\mathrm{D}^4 + a_2\,\mathrm{D}^2 - a_3)\bar{e} = 0, \tag{31}$$
$$(\mathrm{D}^6 - a_1\,\mathrm{D}^4 + a_2\,\mathrm{D}^2 - a_3)\bar{C} = 0. \tag{32}$$

Equation 30 can be factorized as

$$(\mathrm{D}^2 - k_1^2)(\mathrm{D}^2 - k_2^2)(\mathrm{D}^2 - k_3^2)\bar{\theta} = 0, \tag{33}$$

where k_1, k_2, and k_3 are the roots with positive real parts of the characteristic equation,

$$k^6\; - \;a_1\,k^4\; + \;a_2\,k^2\; - a_3\; = \;0. \tag{34}$$

The solution of Eq. 33 has the form,

$$\bar{\theta}(x, s) = \sum_{i=1}^{3}\left(A_i \mathrm{e}^{-k_i\,x}\; + \;B_i \mathrm{e}^{k_i\,x}\right), \tag{35}$$

where $A_i = A_i(s)$ and $B_i = B_i(s)$ are parameters depending on s only.

Similarly, the solution of Eqs. 31 and 32 can be written as

$$\bar{e}(x, s) = \sum_{i=1}^{3}\left(A'_i \mathrm{e}^{-k_i\,x}\; + \;B'_i \mathrm{e}^{k_i\,x}\right), \tag{36}$$
$$\bar{C}(x, s) = \sum_{i=1}^{3}\left(A''_i \mathrm{e}^{-k_i\,x}\; + \;B''_i \mathrm{e}^{k_i\,x}\right), \tag{37}$$

where A'_i, B'_i, A''_i, and B''_i are parameters depending only on s.

Substituting from Eqs. 35–37 into Eqs. 25, 26, and 29, we get

$$A'_i = \frac{k_i^2[k_i^2 - (1 - \varepsilon\,\alpha_1)(s + \tau_0\,s^2)]}{\varepsilon\,(s + \tau_0\,s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1\,s^2]}A_i, \tag{38a}$$
$$B'_i = \frac{k_i^2[k_i^2 - (1 - \varepsilon\,\alpha_1)(s + \tau_0\,s^2)]}{\varepsilon\,(s + \tau_0\,s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1\,s^2]}B_i, \tag{38b}$$

$$A_i'' = \frac{k_i^4 - k_i^2[s^2 + (\varepsilon + 1)(s + \tau_0 s^2)] + s^3(1 + \tau_0 s)}{\varepsilon\,(s + \tau_0 s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} A_i, \tag{39a}$$

$$B_i'' = \frac{k_i^4 - k_i^2[s^2 + (\varepsilon + 1)(s + \tau_0 s^2)] + s^3(1 + \tau_0 s)}{\varepsilon\,(s + \tau_0 s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} B_i. \tag{39b}$$

We thus have

$$\bar{e}(x, s) = \sum_{i=1}^{3} \frac{k_i^2[k_i^2 - (1 - \varepsilon\,\alpha_1)(s + \tau_0 s^2)]}{\varepsilon\,(s + \tau_0 s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} \left(A_i \mathrm{e}^{-k_i x} + B_i \mathrm{e}^{k_i x}\right), \tag{40}$$

$$\bar{C}(x, s) = \sum_{i=1}^{3} \frac{k_i^4 - k_i^2[s^2 + (\varepsilon + 1)(s + \tau_0 s^2)] + s^3(1 + \tau_0 s)}{\varepsilon\,(s + \tau_0 s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} \times \left(A_i \mathrm{e}^{-k_i x} + B_i \mathrm{e}^{k_i x}\right). \tag{41}$$

Integrating both sides of Eq. 7, we obtain upon using the relation, Eq. 40,

$$\bar{u}(x, s) = \sum_{i=1}^{3} \frac{k_i[k_i^2 - (1 - \varepsilon\,\alpha_1)(s + \tau_0 s^2)]}{\varepsilon\,(s + \tau_0 s^2)[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} \left(-A_i \mathrm{e}^{-k_i x} + B_i \mathrm{e}^{k_i x}\right). \tag{42}$$

Substituting from Eqs. 35, 40, and 41 into Eqs. 27a and 28, we get

$$\bar{\sigma}_{xx}(x, s) = \frac{s}{\varepsilon\,(1 + \tau_0 s)} \sum_{i=1}^{3} \frac{[k_i^2 - (1 - \varepsilon\,\alpha_1)(s + \tau_0 s^2)]}{[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} \left(A_i \mathrm{e}^{-k_i x} + B_i \mathrm{e}^{k_i x}\right), \tag{43}$$

$$\bar{P}(x, s) = \frac{\alpha_2\,(1 + \tau s)}{\varepsilon\,(1 + \tau_0 s)} \sum_{i=1}^{3} \frac{k_i^4 - k_i^2[s^2 + (\varepsilon + 1)(s + \tau_0 s^2)] + s^3(1 + \tau_0 s)}{k_i^2[(1 + \alpha_1)\,k_i^2 - \alpha_1 s^2]} \times \left(A_i \mathrm{e}^{-k_i x} + B_i \mathrm{e}^{k_i x}\right). \tag{44}$$

In order to evaluate the unknown parameters, we shall use the Laplace transform of the boundary conditions, Eqs. 21–23, together with Eqs. 35, 43, and 44. We, thus, arrive at the following set of linear equations:

$$\sum_{i=1}^{3} \frac{[k_i^2 - (1-\varepsilon\,\alpha_1)(s+\tau_0\,s^2)]}{[(1+\alpha_1)\,k_i^2 - \alpha_1\,s^2]} \left(A_i \mathrm{e}^{-k_i\,h} + B_i \mathrm{e}^{k_i\,h}\right) = 0, \tag{45}$$

$$\sum_{i=1}^{3} \left(A_i \mathrm{e}^{-k_i\,h} + B_i \mathrm{e}^{k_i\,h}\right) = \bar{f}_1(s), \tag{46}$$

$$\sum_{i=1}^{3} \frac{k_i^4 - k_i^2[s^2 + (\varepsilon+1)(s+\tau_0\,s^2)] + s^3(1+\tau_0 s)}{k_i^2[(1+\alpha_1)\,k_i^2 - \alpha_1\,s^2]} \left(A_i \mathrm{e}^{-k_i\,h} + B_i \mathrm{e}^{k_i\,h}\right) = \frac{\bar{f}_2(s)\,\varepsilon\,(1+\tau_0 s)}{\alpha_2(1+\tau s)}, \tag{47}$$

$$\sum_{i=1}^{3} k_i \left(-A_i \mathrm{e}^{k_i\,h} + B_i \mathrm{e}^{-k_i\,h}\right) = 0, \tag{48}$$

$$\sum_{i=1}^{3} \frac{k_i[k_i^2 - (1-\varepsilon\,\alpha_1)(s+\tau_0\,s^2)]}{(1+\alpha_1)\,k_i^2 - \alpha_1\,s^2} \left(-A_i \mathrm{e}^{k_i\,h} + B_i \mathrm{e}^{-k_i\,h}\right) = 0, \tag{49}$$

$$\sum_{i=1}^{3} \frac{k_i^4 - k_i^2[s^2 + (\varepsilon+1)(s+\tau_0\,s^2)] + s^3(1+\tau_0 s)}{k_i^2[(1+\alpha_1)\,k_i^2 - \alpha_1\,s^2]} \times \left(A_i \mathrm{e}^{k_i\,h} + B_i \mathrm{e}^{-k_i\,h}\right) = 0. \tag{50}$$

Solving the linear system of Eqs. 45–50, we can obtain the parameters $A_1 - A_3$ and $B_1 - B_3$. This completes the solution of the problem in the Laplace transform domain.

4 Inversion of the Laplace Transform

We shall now outline the method used to invert the Laplace transforms in the above equations. Let $\bar{f}(x, s)$ be the Laplace transform of a function $f(x, t)$. The inversion formula for Laplace transforms can be written as [22]

$$f(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{st}\,\bar{f}(x,s)\mathrm{d}s,$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(s)$.

Taking $s = c + \mathrm{i}y$, the above integral takes the form,

$$f(x,t) = \frac{\mathrm{e}^{ct}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ty}\ \bar{f}(x, c+\mathrm{i}y)\mathrm{d}y.$$

Expanding the function $h(x, t) = \exp(-ct)\, f(x, t)$ in a Fourier series in the interval [0,2T], we obtain the approximate formula [23]

$$f(x,t) = f_\infty(x,t) + E_\mathrm{D},$$

where

$$f_\infty(x,t) = \tfrac{1}{2}c_0(x,t) + \sum_{k=1}^{\infty} c_k(x,t) \quad \text{for} \quad 0 \le t \le 2T, \tag{51}$$

and

$$c_k(x,t) = \frac{\mathrm{e}^{ct}}{T} Re\, [\mathrm{e}^{\mathrm{i}k\pi t/T}\ \bar{f}(x, c + \mathrm{i}k\pi/T)], \quad k = 0, 1, 2, \ldots \tag{52}$$

E_D, the discretization error, can be made arbitrarily small by choosing the constant d large enough [23].

As the infinite series in Eq. 51 can only be summed up to a finite number N of terms, the approximate value of $f(x, t)$ becomes

$$f_N(x,t) = \tfrac{1}{2}c_0(x,t) + \sum_{k=1}^{N} c_k(x,t) \quad \text{for} \quad 0 \le t \le 2T. \tag{53}$$

Using the above formula to evaluate $f(x, t)$, we introduce a truncation error, E_T, that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the “Korrecktur” method [23] is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function $f(x,t)$:

$$f(x,t) = f_\infty(x,t) - \mathrm{e}^{-2cT} f_\infty(x, 2T + t) + E'_\mathrm{D},$$

where the discretization error $\left|E'_\mathrm{D}\right| << |E_\mathrm{D}|$ [23]. Thus, the approximate value of $f(x,t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) - \mathrm{e}^{-2cT} f_{N'}(x, 2T + t), \tag{54}$$

N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. 53. Let N be an odd natural number, and let

$$s_m(x,t) = \sum_{k=1}^{m} c_k(x,t)$$

be the sequence of partial sums of Eq. 53. We define the ε-sequence by

$$\varepsilon_{0,m} = 0, \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + 1/(\varepsilon_{p,m+1} - \varepsilon_{p,m}), \quad p = 1, 2, 3, \ldots$$

It can be shown that [23] the sequence,

$$\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$$

converges to $f(x,t) + E_{\mathrm{D}} - c_0/2$ faster than the sequence of partial sums,

$$s_m, m = 1, 2, 3, \ldots$$

The actual procedure used to invert the Laplace transforms consists of using Eq. 54 together with the ε-algorithm. The values of c and T are chosen according to the criteria outlined in [23].

5 Numerical Results

For the purpose of numerical illustration, the problem was solved for the following choice of the functions $f_1(t)$ and $f_2(t)$:

$$f_1(t) = \theta_0 H(t),$$
$$f_2(t) = P_0 H(t),$$

where θ_0 and P_0 are constants and $H(t)$ is the Heaviside unit step function.

We, thus, have

$$\bar{f}_1(s) = \frac{\theta_0}{s},$$
$$\bar{f}_2(s) = \frac{P_0}{s}.$$

The roots k_1 , k_2, and k_3 of the characteristic equation are given by

$$k_1 = \sqrt{\frac{1}{3}[2p\sin q + a_1]}\ ,$$
$$k_2 = \sqrt{\frac{1}{3}[a_1 - p(\sqrt{3}\cos q + \sin q)]}\ ,$$
$$k_3 = \sqrt{\frac{1}{3}[a_1 + p(\sqrt{3}\cos q - \sin q)]}\ ,$$

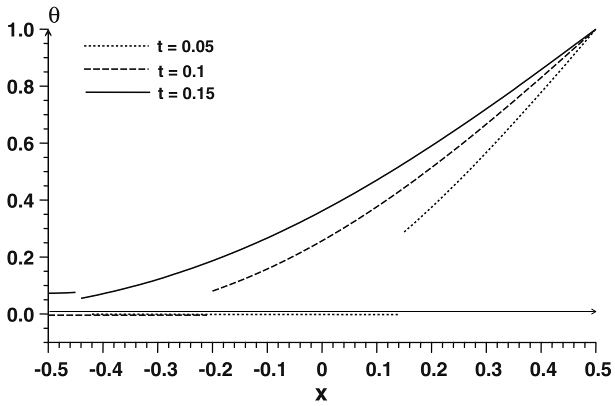

Fig. 1 Temperature distribution

where

$$p = \sqrt{(a_1^2 - 3\,a_2)}, \quad q = \frac{\sin^{-1}(r)}{3}, \quad \text{and} \quad r = -\frac{2\,a_1^3 - 9\,a_1\,a_2 + 27\,a_3}{2\,p^3}.$$

Copper material was chosen for purposes of numerical evaluations. The material constants of the problem are thus given by in SI units [24]:

$T_0 = 293\,\mathrm{K}$, $\rho = 8{,}954\,\mathrm{kg\cdot m^{-3}}$, $\tau_0 = 0.02$ s, $\tau = 0.2$ s,
$c_E = 383.1\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, $\alpha_t = 1.78 \times 10^{-5}\,\mathrm{K^{-1}}$, $k = 386\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$,
$\lambda = 7.76 \times 10^{10}\,\mathrm{kg\cdot m^{-1}\cdot s^{-2}}$, $\mu = 3.86 \times 10^{10}\,\mathrm{kg\cdot m^{-1}\cdot s^{-2}}$,
$\alpha_c = 1.98 \times 10^{-4}\,\mathrm{m^3\cdot kg^{-1}}$, $d = 0.85 \times 10^{-8}\,\mathrm{kg\cdot s\cdot m^{-3}}$,
$a = 1.2 \times 10^4\,\mathrm{m^2\cdot s^{-2}\cdot K^{-1}}$, $b = 0.9 \times 10^6\,\mathrm{m^5\cdot kg^{-1}\cdot s^{-2}}$.

Using these values, it was found that

$$\eta = 8886.73,\ \varepsilon = 0.0168,\ \beta^2 = 4,\ \alpha_1 = 5.43,\ \alpha_2 = 0.533,\ \text{and}\ \alpha_3 = 36.24.$$

It should be noted that a unit of dimensionless time corresponds to 6.5×10^{-12} s, while a unit of dimensionless length corresponds to 2.7×10^{-8} m.

The computations were carried out for three values of dimensionless time, namely, for $t = 0.05$, $t = 0.1$, and $t = 0.15$. The temperature, displacement, stress, concentration, and chemical potential are shown in Figs. 1, 2, 3, 4, and 5, respectively. Dotted lines represent the case when $t = 0.05$, dashed lines represent the case when $t = 0.1$, while solid lines represent the case when $t = 0.15$.

As expected from the order of the partial differential equation, we have three waves emanating from each surface; the fronts of these waves are depicted in the figures as discontinuities in the functions in Figs. 1 and 3–5 or in the first derivative in Fig. 2 because the displacement is a continuous function. Of course, some of these discontinuities are very small to show in the figures.

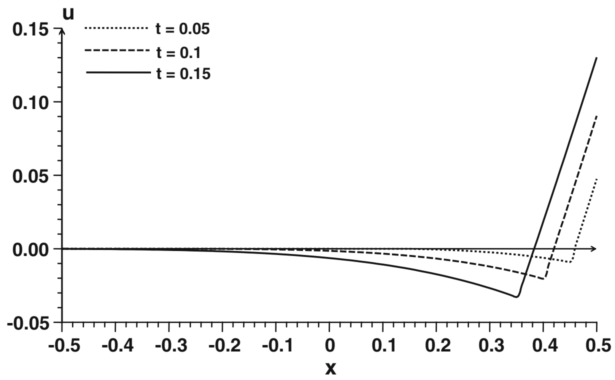

Fig. 2 Displacement distribution

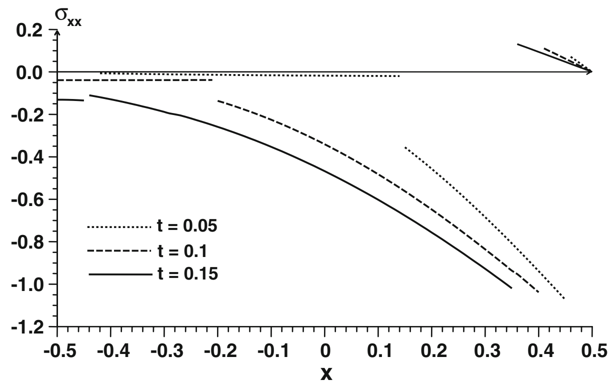

Fig. 3 Stress distribution

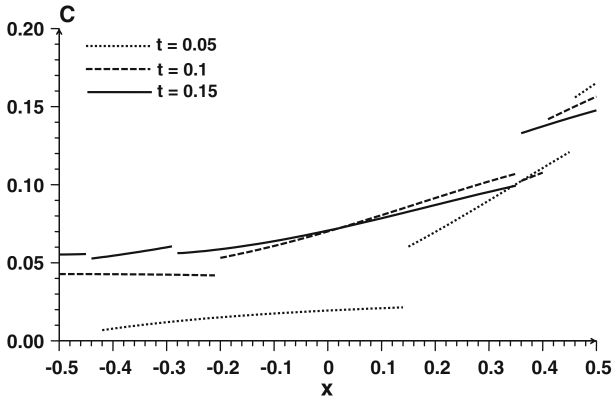

Fig. 4 Concentration distribution

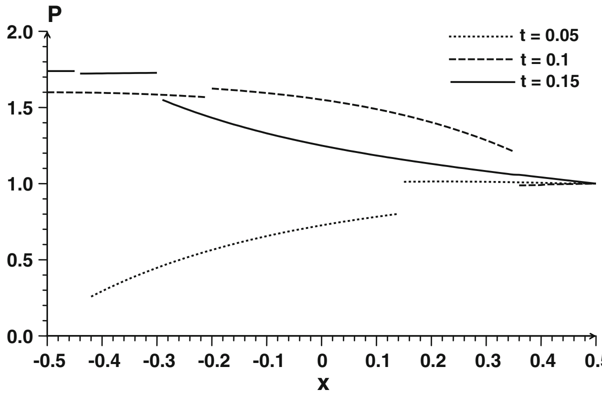

Fig. 5 Chemical potential distribution

Table 1 Location of wave fronts

t	Location of wave fronts		
0.05	−0.428	0.149	0.451
0.1	−0.202	0.355 Reflected	0.402
0.15	−0.447	−0.283 Reflected twice	0.353

It was found that there are three waves with dimensionless speeds of

$$\nu_1 = 0.98, \nu_2 = 7.02, \nu_3 = 18.55.$$

The locations of the fronts of these waves are shown in Table 1 for different values of time.

It is clear from the graphs that the effect of diffusion on the temperature and displacement is very weak but has a noticeable effect on the stress.

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